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TRANSIENT NON-AXISYMMETRIC WAVE PROPAGATION IN AN INFINITE ISOTROPIC ELASTIC PLATE*

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Abstract—Using linear elasticity theory, the formal solutions to a general class of non-axisymmetric, transient, elastic wave-propagation problems involving an infinite, isotropic, elastic plate are given. For the particular problem of the sudden application of normal half-ring loads on the faces of the plate, explicit inversion is achieved, and some far-field numerical results, based on a low-frequency, large-wavelength approximation are presented.

INTRODUCTION

THE present paper gives the formal (transformed) solutions, based on linear elasticity theory, to a general class of non-axisymmetric, transient, wave-propagation problems involving an infinite, isotropic, elastic plate. The theory is applied to the particular problem of the sudden application of normal half-ring loads on the faces of the plate, and for this case explicit inversion is achieved. Some far-field numerical results, based on a low-frequency, large-wavelength approximation, are presented.

Recent years have seen the emergence of greatly increased interest in transient elastic wave propagation in isotropic rods and plates. Axially symmetric problems involving the rod geometry have been treated by Skalak [1] and by Fox *et al.* ([2] and [3]). Extensions of this type of work to non-axisymmetric cases were given by De Vault and Curtis [4], whose studies more or less completed the treatment of the mixed problem for semi-infinite rods. The plate geometry has also received attention, work on axisymmetric problems being done by Fulton and Sneddon [5], Miklowitz [6] and Knopoff and Gilbert [7]. However, the only contributions on non-axisymmetric inputs that appear to exist are those given by Davids *et al.* ([8] and [9]), who treated specialized cases involving simple, trigonometric, angular dependencies. Mention should also be made of the work of Harkrider [10], who considered waves generated by a time-harmonic, non-axisymmetric source, buried in a layered half-space with a stress-free surface. Thus, as far as the authors are aware, there has been no general treatment of non-axisymmetric, transient, elastic, wave propagation in a plate, and it was this basic lack of information which led to interest in the present work.

The formal solutions are obtained by means of multi-integral transforms (Laplace and Fourier), and a superposition method based on inverse Hankel transforms. Inversion,

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for the particular case of the sudden application of normal, half-ring loads on the plate surfaces, is carried out by means of residue theory, and some far-field information is obtained by means of a low-frequency, large-wavelength approximation.

STATEMENT OF GENERAL PROBLEM AND DERIVATION OF FORMAL SOLUTIONS

The problem to be treated is that of an infinite, homogeneous, isotropic, elastic plate of thickness $2H$, on whose surfaces certain time-dependent stresses, which generate non-axisymmetric waves, are applied. Cylindrical coordinates r, θ, and z, are used, with the z-axis—origin at the plate center—perpendicular to the plate surfaces.

The displacement equation of motion of a linear homogeneous, isotropic, elastic solid is, for zero body forces,

$$(\lambda+2\mu)\nabla(\nabla\,.\,\mathbf{u})-\mu\nabla\times\nabla\times\mathbf{u} = \rho'\frac{\partial^2\mathbf{u}}{\partial t^2} \tag{1}$$

where $\mathbf{u}$ is the displacement vector, λ and μ are Lamé's constants, ρ' is the material density, and t is the time. Equation (1) is satisfied by

$$\mathbf{u} = \nabla\varphi+\nabla\times\mathbf{A} \tag{2}$$

provided

$$\nabla^2\varphi = \frac{1}{c_d^2}\frac{\partial^2\varphi}{\partial t^2} \tag{3}$$

$$\nabla^2\mathbf{A} = \frac{1}{c_s^2}\frac{\partial^2\mathbf{A}}{\partial t^2} \tag{4}$$

where $c_d^2 = (\lambda+2\mu)/\rho'$ and $c_s^2 = \mu/\rho'$ are the dilatational and equivoluminal wave speeds squared, respectively. In cylindrical coordinates, equation (4) is satisfied by

$$\mathbf{A} = \chi\mathbf{e}_z+\nabla\times(\psi\mathbf{e}_z) \tag{5}$$

where $\mathbf{e}_z$ is a unit vector along the cylindrical axis, provided the scalar functions χ and ψ satisfy

$$\nabla^2\chi = \frac{1}{c_s^2}\frac{\partial^2\chi}{\partial t^2} \tag{6}$$

$$\nabla^2\psi = \frac{1}{c_s^2}\frac{\partial^2\psi}{\partial t^2}. \tag{7}$$

It can readily be shown that χ and ψ represent SH, or torsional, and SV type waves, respectively.

Substituting equation (5) into equation (2), and using equation (7), the displacement–potential relations are found to be

$$u_r = \frac{\partial\varphi}{\partial r}+\frac{1}{r}\frac{\partial\chi}{\partial\theta}+\frac{\partial^2\psi}{\partial r\partial z} \tag{8}$$

$$u_\theta = \frac{1}{r}\frac{\partial\varphi}{\partial\theta}-\frac{\partial\chi}{\partial r}+\frac{1}{r}\frac{\partial^2\psi}{\partial\theta\,\partial z} \tag{9}$$

$$u_z = \frac{\partial\varphi}{\partial z}+\frac{\partial^2\psi}{\partial z^2}-\frac{1}{c_s^2}\frac{\partial^2\psi}{\partial t^2}. \tag{10}$$

The pertinent stress–displacement relations are

$$\sigma_{zz} = \lambda\left[\frac{1}{r}\frac{\partial}{\partial r}(ru_r)+\frac{1}{r}\frac{\partial u_\theta}{\partial \theta}+\frac{\partial u_z}{\partial z}\right]+2\mu\frac{\partial u_z}{\partial z} \tag{11}$$

$$\sigma_{z\theta} = \mu\left(\frac{1}{r}\frac{\partial u_z}{\partial \theta}+\frac{\partial u_\theta}{\partial z}\right) \tag{12}$$

$$\sigma_{zr} = \mu\left(\frac{\partial u_r}{\partial z}+\frac{\partial u_z}{\partial r}\right). \tag{13}$$

The boundary conditions, which correspond to a fairly general class of non-axisymmetric problems, are taken as

$$\sigma_{zz} = -f_1(r)g_1(\theta)h_1(t), \qquad z = \pm H \tag{14}$$

$$\sigma_{zr} = \pm f_2(r)g_2(\theta)h_2(t), \qquad z = \pm H \tag{15}$$

$$\sigma_{z\theta} = \mp f_2(r)g_3(\theta)h_2(t), \qquad z = \pm H. \tag{16}$$

It should be noted that these boundary conditions are symmetric w.r.t. the mid-plane of the plate, so that only the corresponding wave motions are generated (i.e. compressional and a class of torsional modes). However, this restriction to such modes, which is taken for algebraic convenience, in no way detracts from the generality of the subsequent solution technique.

Taking the Laplace and finite Fourier cosine transforms (w.r.t. t and θ, respectively) of equations (3), (7), (8), (10), (11), and (13), and the Laplace and finite Fourier sine transform of equations (6), (9), and (12), gives, on assuming zero initial conditions,

$$\frac{\partial^2\tilde{\bar{\phi}}}{\partial r^2}+\frac{1}{r}\frac{\partial\tilde{\bar{\phi}}^{C}}{\partial r}-\left(\frac{\beta^2}{r^2}+\frac{p^2}{c_d^2}\right)\tilde{\bar{\phi}}^{C}+\frac{\partial^2\tilde{\bar{\phi}}^{C}}{\partial z^2} = 0 \tag{17}$$

$$\frac{\partial^2\tilde{\bar{\psi}}^{C}}{\partial r^2}+\frac{1}{r}\frac{\partial\tilde{\bar{\psi}}^{C}}{\partial r}-\left(\frac{\beta^2}{r^2}+\frac{p^2}{c_s^2}\right)\tilde{\bar{\psi}}^{C}+\frac{\partial^2\tilde{\bar{\psi}}^{C}}{\partial z^2} = 0 \tag{18}$$

$$\frac{\partial^2\tilde{\bar{\chi}}^{S}}{\partial r^2}+\frac{1}{r}\frac{\partial\tilde{\bar{\chi}}^{S}}{\partial r}-\left(\frac{\beta^2}{r^2}+\frac{p^2}{c_s^2}\right)\tilde{\bar{\chi}}^{S}+\frac{\partial^2\tilde{\bar{\chi}}^{S}}{\partial z^2} = 0 \tag{19}$$

$$\tilde{\bar{u}}_r^{C} = \frac{\partial\tilde{\bar{\phi}}^{C}}{\partial r}+\frac{\beta}{r}\tilde{\bar{\chi}}^{S}+\frac{\partial^2\tilde{\bar{\psi}}^{C}}{\partial r\,\partial z} \tag{20}$$

$$\tilde{\bar{u}}_z^{C} = \frac{\partial\tilde{\bar{\phi}}^{C}}{\partial z}+\frac{\partial^2\tilde{\bar{\psi}}^{C}}{\partial z^2}-\frac{p^2}{c_s^2}\tilde{\bar{\psi}}^{C} \tag{21}$$

$$-\tilde{\bar{u}}_\theta^{S} = \frac{\beta}{r}\tilde{\bar{\phi}}^{C}+\frac{\partial\tilde{\bar{\chi}}^{S}}{\partial r}+\frac{\beta}{r}\frac{\partial\tilde{\bar{\psi}}^{C}}{\partial z} \tag{22}$$

$$\tilde{\bar{\sigma}}_{zz}^{C} = \lambda\frac{p^2}{c_d^2}\tilde{\bar{\phi}}^{C}+2\mu\frac{\partial\tilde{\bar{u}}_z^{C}}{\partial z} \tag{23}$$

$$\tilde{\bar{\sigma}}_{zr}^{C} = \mu\left(\frac{\partial\tilde{\bar{u}}_r^{C}}{\partial z}+\frac{\partial\tilde{\bar{u}}_z^{C}}{\partial r}\right) \tag{24}$$

$$\tilde{\bar{\sigma}}_{z\theta}^{S} = \mu\left(-\frac{\beta}{r}\tilde{\bar{u}}_z^{C}+\frac{\partial\tilde{\bar{u}}_\theta^{S}}{\partial z}\right) \tag{25}$$

where a bar denotes the Laplace transform, parameter p, and a tilde denotes a Fourier transform, parameter β. The superscripts C and S denote cosine and sine transforms, respectively. The θ dependence could also be suppressed by interchanging the roles of the finite Fourier transforms, and for more general problems the expressions obtained by both procedures should be superposed. The assumption here is that the θ-dependence in the boundary conditions is such that u_r is an even function of θ, etc.

Equations (17) through (25) are partial differential equations, and, in the scheme of multi-integral transform approaches, another independent variable must be suppressed. Now it is known that boundary conditions are "built into" an integral transform, and that these boundary conditions are of the mixed type. Thus z in the above equations cannot be suppressed by means of an integral transform, since the boundary conditions on $z = \pm H$ are of the non-mixed type. Hence one has to work with r, and a natural procedure would seem to be to try and use Hankel transforms. However, no Hankel transforms which would achieve this appear to exist (for reasons which will be discussed later) and, following Harkrider [10], r is suppressed here by means of a superposition technique. The expressions

$$\tilde{\bar{\varphi}}^{C} = \int_0^\infty Z_1(k, p, z) J_\beta(kr)\, dk \tag{26}$$

$$\tilde{\bar{\psi}}^{C} = \int_0^\infty Z_2(k, p, z) J_\beta(kr)\, dk \tag{27}$$

$$\tilde{\bar{\chi}}^{S} = \int_0^\infty Z_3(k, p, z) J_\beta(kr)\, dk \tag{28}$$

where J denotes a Bessel function of the first kind, satisfy equations (17), (18), and (19), provided

$$\frac{d^2 Z_1}{dz^2} - \eta_2^2 Z_1 = 0 \tag{29}$$

$$\frac{d^2 Z_j}{dz^2} - \eta_1^2 Z_j = 0, \qquad j = 1, 2, \tag{30}$$

where

$$\eta_2^2 = (k^2 + p^2/c_d^2), \qquad \eta_1^2 = (k^2 + p^2/c_s^2).$$

The solutions to equations (29) and (30) which are symmetric w.r.t. the mid-plane of the plate are

$$Z_1 = B_1(k, p) \cosh \eta_2 z, \qquad Z_2 = B_2(k, p) \sinh \eta_1 z, \qquad Z_3 = B_3(k, p) \cosh \eta_1 z$$

where B_1, B_2, and B_3 are to be determined. Substituting these expressions into equations (26), (27), and (28), gives the potentials, and once these are known, the displacements and stresses can be found from equations (20) through (25). The stresses are found to be

$$\tilde{\bar{\sigma}}_{zz}^{C} = \mu \int_0^\infty [B_1(\eta_1^2 + k^2) \cosh \eta_2 z + 2B_2 \eta_1 k^2 \cosh \eta_1 z] J_\beta(kr)\, dk \tag{31}$$

$$\begin{aligned} \frac{1}{\mu} \tilde{\bar{\sigma}}_{zr}^{C} = & \int_0^\infty [2B_1 \eta_2 \sinh \eta_2 z + B_2(\eta_1^2 + k^2) \sinh \eta_1 z] \frac{d}{dr} J_\beta(kr)\, dk \\ & + \int_0^\infty B_3 \eta_1 \sinh \eta_1 z \frac{\beta}{r} J_\beta(kr)\, dk \end{aligned} \tag{32}$$

$$-\frac{1}{\mu}\tilde{\bar{\sigma}}^S_{z\theta} = \int_0^\infty [2B_1\eta_2 \sinh \eta_2 z + B_2(\eta_1^2+k^2)\sinh \eta_1 z]\frac{\beta}{r}J_\beta(kr)\,\mathrm{d}k + \int_0^\infty B_3\eta_1 \sinh \eta_1 z\frac{\mathrm{d}}{\mathrm{d}r}J_\beta(kr)\,\mathrm{d}k. \tag{33}$$

One comment seems appropriate here. It was mentioned previously that no Hankel transforms which are directly applicable to the present problem appear to exist. The reasons for this can be seen, for instance, in equation (33). There the functions $(\beta/r)J_\beta(kr)$ and $(\mathrm{d}/\mathrm{d}r)J_\beta(kr)$ are each multiplied by different, arbitrary functions of k, and the kernel of any Hankel transform would have to reflect this fact.

The above equations are in the form of integral superpositions over the wavenumber k, whereas the boundary conditions—Laplace and finite Fourier transformed—still contain arbitrary functions of r. Thus, representations of these arbitrary functions, which are suited for application to equations (31), (32), and (33), must be found. This key feature of the solution procedure is achieved here by means of *inverse* Hankel transforms. Focusing on equation (15), for example, $f_2(r)$ may be written

$$f_2(r) = \int_0^\infty k\hat{f}_2^{\beta-1}(k)J_{\beta-1}(kr)\,\mathrm{d}k = \int_0^\infty \hat{f}_2^{\beta-1}(k)\left[\frac{\beta}{r}J_\beta(kr)+\frac{\mathrm{d}}{\mathrm{d}r}J_\beta(kr)\right]\mathrm{d}k$$

where

$$\hat{f}_2^{\beta-1}(k) = \int_0^\infty rf_2(r)J_{\beta-1}(kr)\,\mathrm{d}r.$$

Using representations such as these, the transformed boundary conditions can then be applied to equations (31), (32), and (33). On noting the linear independence of $(\beta/r)J_\beta(kr)$ and $(\mathrm{d}/\mathrm{d}r)J_\beta(kr)$, one finds that the system is overdetermined,* in that five equations are obtained to evaluate B_1, B_2, and B_3. However, the system is determinate, if the condition

$$\tilde{\bar{g}}_2^C(\beta) = \tilde{\bar{g}}_3^S(\beta)$$

is imposed, and henceforth this is assumed to be satisfied. Then B_1, B_2, and B_3, and consequently the transformed field quantities, can be found. For example, the transformed displacements are

$$\mu\tilde{\bar{u}}_r^C = \int_0^\infty \frac{F_1(k,p,z)}{D(k,p)}\frac{\mathrm{d}}{\mathrm{d}r}J_\beta(kr)\,\mathrm{d}k + \int_0^\infty \frac{\Gamma_2(k,\beta,p)\cosh \eta_1 z}{\eta_1 \sinh \eta_1 H}\frac{\beta}{r}J_\beta(kr)\,\mathrm{d}k \tag{34}$$

$$\mu\tilde{\bar{u}}_z^C = \int_0^\infty \frac{F_2(k,p,z)}{D(k,p)}J_\beta(kr)\,\mathrm{d}k \tag{35}$$

$$-\mu\tilde{\bar{u}}_\theta^S = \int_0^\infty \frac{F_1(k,p,z)}{D(k,p)}\frac{\beta}{r}J_\beta(kr)\,\mathrm{d}k + \int_0^\infty \frac{\Gamma_2(k,\beta,p)\cosh \eta_1 z}{\eta_1 \sinh \eta_1 H}\frac{\mathrm{d}}{\mathrm{d}r}J_\beta(kr)\,\mathrm{d}k \tag{36}$$

* Presumably the solutions obtained when the roles of the finite Fourier transforms are interchanged must also be considered.

where

$$F_1(k, p, z) = \eta_1[\Gamma_2(\eta_1^2+k^2)\cosh\eta_2 H+2\Gamma_1\eta_2\sinh\eta_2 H]\cosh\eta_1 z$$
$$-[\Gamma_1(\eta_1^2+k^2)\sinh\eta_1 H+2\Gamma_2\eta_1 k^2\cosh\eta_1 H]\cosh\eta_2 z$$
$$F_2(k, p, z) = k^2[\Gamma_2(\eta_1^2+k^2)\cosh\eta_2 H+2\Gamma_1\eta_2\sinh\eta_2 H]\sinh\eta_1 z$$
$$-\eta_2[\Gamma_1(\eta_1^2+k^2)\sinh\eta_1 H+2\Gamma_2\eta_1 k^2\cosh\eta_1 H]\sinh\eta_2 z$$
$$D(k, p) = (\eta_1^2+k^2)^2\cosh\eta_2 H\sinh\eta_1 H-4k^2\eta_1\eta_2\cosh\eta_1 H\sinh\eta_2 H$$
$$\Gamma_1(k, \beta, p) = k\hat{f}_1^{\beta}(k)\tilde{g}_1^C(\beta)\bar{h}_1(p)$$
$$\Gamma_2(k, \beta, p) = \hat{f}_2^{\beta-1}(k)\tilde{g}(\beta)\bar{h}_2(p)$$
$$\tilde{g}(\beta) = \tilde{g}_2^C(\beta) = \tilde{g}_3^S(\beta).$$

Equations (34), (35), and (36) constitute the formal solutions, but rather than approach inversion in a general context, attention is now focused on a particular problem.

SUDDEN APPLICATION OF NORMAL HALF-RING LOADS

In this section the general theory is applied to the particular problem of the sudden, symmetric, application of normal half-ring loads. This particular excitation is of interest in that the angular dependence involves more than simple trigonometric functions, and it is capable of being simulated experimentally by ring transducers.

The boundary conditions are (see Fig. 1)

$$\sigma_{zz} = -\frac{\sigma_0\delta(r-a)}{\pi r}Q(\theta)\mathrm{H}(t), \qquad z = \pm H \tag{37}$$

$$\sigma_{zr} = \sigma_{z\theta} = 0, \qquad z = \pm H \tag{38}$$

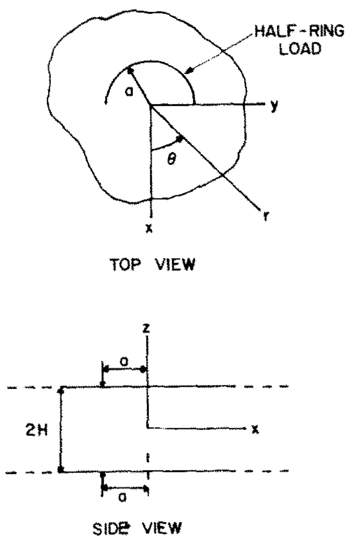

FIG. 1. Geometry of the half-ring load.

where δ denotes the delta function, H(t) the Heaviside unit step function, σ_0 is a load constant, and

$$Q(\theta) = \begin{cases} 0, & -\pi/2 < \theta < \pi/2 \\ 1, & \pi/2 < \theta < \pi, \ -\pi < \theta < -\pi/2. \end{cases}$$

Substituting equations (37) and (38) into equations (34), (35), and (36), the transformed displacements are found to be

$$\tilde{\bar{u}}_r^C = -\frac{\sigma_0\gamma(\beta)}{\pi\mu}\int_0^\infty \frac{kF_1(k,p,z)}{pD(k,p)} J_\beta(ka)\frac{\mathrm{d}}{\mathrm{d}r}J_\beta(kr)\,\mathrm{d}k \tag{39}$$

$$\tilde{\bar{u}}_z^C = \frac{\sigma_0\gamma(\beta)}{\pi\mu}\int_0^\infty \frac{kF_2(k,p,z)}{pD(k,p)} J_\beta(ka)J_\beta(kr)\,\mathrm{d}k \tag{40}$$

$$\tilde{\bar{u}}_\theta^S = \frac{\sigma_0\gamma(\beta)}{\pi\mu}\int_0^\infty \frac{kF_1(k,p,z)}{pD(k,p)} J_\beta(ka)\frac{\beta}{r}J_\beta(kr)\,\mathrm{d}k \tag{41}$$

where now

$$F_1(k,p,z) = (\eta_1^2+k^2)\sinh\eta_1 H\cosh\eta_2 z - 2\eta_1\eta_2\sinh\eta_2 H\cosh\eta_1 z$$

$$F_2(k,p,z) = 2k^2\eta_2\sinh\eta_2 H\sinh\eta_1 z - \eta_2(\eta_1^2+k^2)\sinh\eta_1 H\sinh\eta_2 z$$

and

$$\gamma(\beta) = \begin{cases} 0, & \beta = 2,4,6,\ldots \\ \pi, & \beta = 0 \\ -\dfrac{2}{\beta}\sin\dfrac{\beta\pi}{2}, & \beta = 1,3,5,\ldots \end{cases} \tag{42}$$

Applying the Fourier inversion* theorems, gives, on using equation (42),

$$\frac{\mu\pi}{\sigma_0}\bar{u}_r = \frac{2}{\pi}\sum_{\beta=0,1,3,}^{\infty}\varepsilon_\beta\left[\int_0^\infty \frac{kF_1(k,p,z)}{pD(k,p)}J_\beta(ka)\frac{\mathrm{d}}{\mathrm{d}r}J_\beta(kr)\,\mathrm{d}k\right]\cos\beta\theta \tag{43}$$

$$\frac{\mu\pi}{\sigma_0}\bar{u}_z = -\frac{2}{\pi}\sum_{\beta=0,1,3,}^{\infty}\varepsilon_\beta\left[\int_0^\infty \frac{kF_2(k,p,z)}{pD(k,p)}J_\beta(ka)J_\beta(kr)\,\mathrm{d}k\right]\cos\beta\theta \tag{44}$$

$$\frac{\mu\pi^2\bar{u}_\theta}{2\sigma_0} = -\frac{1}{r}\sum_{\beta=1,3,}^{\infty}\sin\frac{\beta\pi}{2}\left[\int_0^\infty \frac{kF_1(k,p,z)}{pD(k,p)}J_\beta(ka)J_\beta(kr)\,\mathrm{d}k\right]\sin\beta\theta \tag{45}$$

where

$$\varepsilon_\beta = \begin{cases} -\pi/4, & \beta = 0 \\ \dfrac{1}{\beta}\sin\dfrac{\beta\pi}{2}, & \beta = 1,3,\ldots \end{cases}$$

The next step in the development of the solutions is the inversion of the Laplace transforms. It is now well known that two fruitful methods exist, one giving results suited for near-field computation, whereas the other is geared to the far-field. The method suited to obtaining near-field information (see, for instance, Rosenfeld and Miklowitz [11]), consists

* It should be noted that if the boundary conditions had a cos $\beta\theta$ or sin $\beta\theta$ dependence, then the response would have a cos $\beta\theta$ or sin $\beta\theta$ dependence.

of expressing the various hyperbolic functions in equations (43), (44), and (45), in terms of exponential functions, expanding the denominators, and arranging the results into a single series of exponentials, each of which can be identified with a particular type of wave. Focusing on equation (44), for example, and restricting attention to the line $r = 0$ (with the result that only one term in the β series survives), the contribution from the direct compressional wave can be shown to be, on setting $k = p\omega$,

$$\frac{\mu\pi}{\sigma_0}\bar{u}_z = -\frac{1}{2}\int_0^\infty \frac{\omega\eta_2(\omega)[\omega^2+\eta_1^2(\omega)]}{R(\omega)} J_0(p\omega a)\,e^{-p\eta_2(\omega)(H-z)}\,d\omega \tag{46}$$

where

$$\eta_1(\omega) = (\omega^2+1/c_s^2)^{\frac{1}{2}}, \qquad \eta_2(\omega) = (\omega^2+1/c_d^2)^{\frac{1}{2}}$$
$$R(\omega) = [\omega^2+\eta_1^2(\omega)]^2 - 4\omega^2\eta_1(\omega)\eta_2(\omega).$$

Replacing the Bessel function by an integral representation, and interchanging orders of integration, equation (46) may be written

$$\frac{\mu\pi}{\sigma_0}u_z = -\frac{\mathscr{R}}{\pi}\int_0^1 \frac{1}{(1-\xi^2)^{\frac{1}{2}}}\left\{L^{-1}\int_0^\infty \frac{\omega\eta_2(\omega)[\omega^2+\eta_1^2(\omega)]}{R(\omega)} e^{-p[\eta_2(\omega)(H-z)+i\omega a\xi]}\,d\omega\right\}d\xi$$

where $\mathscr{R}$ denotes real part and L^{-1} represents the inversion integral of the Laplace transform. As can be seen from the work of Rosenfeld and Miklowitz [11], the inversion integral can be written in closed form by means of the Cagniard–DeHoop method. Thus results* in the form of a sum of finite integrals (suited for digital computation) can be obtained, each integral representing the contribution from a particular type of wave. However, this aspect will not be developed any further here.

The method of inversion outlined above is not suited for far-field studies in that a prohibitive number of wave contributions would have to be assessed. For such studies the so-called modal method of inversion is called for, and this will now be pursued at some length. In this approach, inversion is achieved with the aid of residue theory. The integrands in equations (43), (44), and (45) are even functions of η_1 and η_2 and so there are no branch points in the p-plane. Then the only singularities are poles at $p = 0$ and at the zeros of $D(k, p)$, which, as shown by the authors [13], are pure imaginary, simple, and infinite in number. Evaluating the residues and interchanging summation and integration gives

$$\begin{aligned}\frac{\mu H\pi u_r}{\sigma_0} = \frac{2}{\pi}\sum_{\beta=0,1,3,}^{\infty} \varepsilon_\beta\left\{\int_0^\infty \frac{1}{K}G_3(K\zeta)J_\beta(KR)\frac{d}{d\rho}J_\beta(K\rho)\,dK\right\}\cos\beta\theta \\ -\frac{4}{\pi}\sum_{\beta=0,1,3,}^{\infty}\varepsilon_\beta\left\{\sum_{n=1}^{\infty}\int_0^\infty \frac{F_3(K,\Omega_n,\zeta)\cos\Omega_n(K)\tau}{\Omega_n^2(K)M(K,\Omega_n)}J_\beta(KR)\frac{d}{d\rho}J_\beta(K\rho)\,dK\right\}\cos\beta\theta\end{aligned} \tag{47}$$

$$\begin{aligned}\frac{\mu H\pi u_z}{\sigma_0} = -\frac{2}{\pi}\sum_{\beta=0,1,3,}^{\infty}\varepsilon_\beta\left\{\int_0^\infty G_4(K,\zeta)J_\beta(KR)J_\beta(K\rho)\,dK\right\}\cos\beta\theta \\ +\frac{4}{\pi}\sum_{\beta=0,1,3,}^{\infty}\varepsilon_\beta\left\{\sum_{n=1}^{\infty}\int_0^\infty \frac{KF_4(K,\Omega_n,\zeta)\cos\Omega_n(K)\tau}{\Omega_n^2(K)M(K,\Omega_n)}J_\beta(KR)J_\beta(K\rho)\,dK\right\}\cos\beta\theta\end{aligned} \tag{48}$$

* These results are analogous to those obtained by Pekeris [12] in his work on the elastic half-space.

$$\frac{\mu H\pi^2 u_\theta}{2\sigma_0} = -\frac{1}{\rho}\sum_{\beta=1,3,}^{\infty} \sin\frac{\beta\pi}{2}\left\{\int_0^\infty \frac{1}{K} G_3(K\zeta)J_\beta(KR)J_\beta(K\rho)\,\mathrm{d}K\right\}\sin\beta\theta$$
$$+\frac{2}{\rho}\sum_{\beta=1,3,}^{\infty}\sin\frac{\beta\pi}{2}\left\{\sum_{n=1}^{\infty}\int_0^\infty \frac{F_3(K,\Omega_n,\zeta)\cos\Omega_n(K)\tau}{\Omega_n^2 M(K,\Omega_n)} J_\beta(KR)J_\beta(K\rho)\,\mathrm{d}K\right\}\sin\beta\theta \tag{49}$$

where

$$G_3(K\zeta) = \frac{K(\cosh K\cosh K\zeta-\zeta\sinh K\sinh K\zeta)-1/(\alpha^2-1)\sinh K\cosh K\zeta}{2K+\sinh 2K}$$

$$G_4(K\zeta) = \frac{K(\zeta\sinh K\cosh K\zeta-\cosh K\sinh K\zeta)-\alpha^2/(\alpha^2-1)\sinh K\sinh K\zeta}{2K+\sinh 2K}$$

$$F_3(K,\Omega_n,\zeta) = \psi_n\sin Kk_{sn}\cosh Kk_{dn}\zeta-2k_{sn}k_{dn}\sinh Kk_{dn}\cos Kk_{sn}\zeta$$

$$F_4(K,\Omega_n,\zeta) = k_{dn}(2\sinh Kk_{dn}\sin Kk_{sn}\zeta-\psi_n\sin Kk_{sn}\sinh Kk_{dn}\zeta)$$

$$M(K,\Omega_n) = -\left(\frac{\psi_n^2}{k_{sn}}+\frac{4k_{sn}}{\alpha^2}\right)\cos Kk_{sn}\cosh Kk_{dn}+\left(\frac{\psi_n^2}{\alpha^2 k_{dn}}-4k_{dn}\right)\sin Kk_{sn}\sinh Kk_{dn}$$
$$+4\frac{\psi_n}{K}\cosh Kk_{dn}\sin Kk_{sn}-\frac{4}{K}\left(\frac{k_{sn}}{\alpha^2 k_{dn}}-\frac{k_{dn}}{k_{sn}}\right)\cos Kk_{sn}\sinh Kk_{dn}$$

$$\psi_n = 2-\frac{1}{K^2}\Omega_n^2(k);\qquad k_{dn} = \left[1-\frac{\Omega_n^2(K)}{K^2\alpha^2}\right]^{\frac{1}{2}};\qquad k_{sn} = \left[\frac{\Omega_n^2(K)}{K^2}-1\right]^{\frac{1}{2}}.$$

and the following dimensionless variables have been introduced: $HR = a, H\rho = r, H\zeta = z$, $\tau = \omega_s t$, $H\omega_s = c_s$, $\alpha^2 = c_d^2/c_s^2$, $K = kH$, $\omega_s\Omega_n(K) = \omega_n(k)$, where $\omega_n(k)$ is a root of the Rayleigh–Lamb frequency equation for the propagation of harmonic, straight-crested, compressional waves in an infinite plate of thickness $2H$.

The form of the above solutions clearly indicates the greater complexity involved in non-axisymmetric problems over axisymmetric ones. Both have in common the sum over the roots of the Rayleigh–Lamb frequency equation, but in the present problem, lack of axial symmetry leads to an additional Neumann-type summation over Bessel functions. Complicated though the response is, it should be mentioned that it is simpler than the rod response, as given by De Vault and Curtis (4), in that for the rod the sum over the Bessel functions couples into the modes of circumferential propagation, whereas no such modes exist for the plate.

Extraction of numerical information from the solutions as they stand presents a formidable task, but some useful far-field approximations can be made and these will now be described.

FAR-FIELD APPROXIMATIONS

Miklowitz [6] has shown on the basis of the first three modes of the Rayleigh–Lamb frequency equation, that it is the low frequency, large wavelength portion of the lowest mode which governs the earliest arriving disturbance. Though still earlier arriving waves, associated with the dilatational wave speed, can occur, experiment* (see Fox *et al.* [3]

* The experiments have been performed for the rod geometry, but because of the close similarity between the rod and plate spectra, the inference is that they are also descriptive of the plate.

and Miklowitz and Nisewanger [14]) has indicated that these high-frequency components have no appreciable amplitude in the far-field. The assumption here is that the distribution in the θ-direction does not alter this situation, and so henceforth attention is restricted to that portion of the spectrum considered by Miklowitz in [6], i.e., the summations over n in equations (47), (48), and (49) is deleted.

Equation (47) becomes, on using some relations between Bessel functions and their derivatives,

$$\begin{aligned}\frac{\mu H\pi u_r}{\sigma_0} &= \frac{2}{\pi}\sum_{\beta=0,1,3}^{\infty}\varepsilon_\beta\left\{\int_0^\infty G_3(K\zeta)J_\beta(KR)\left[\frac{\beta}{K^2\rho^2}J_\beta(K\rho)-J_{\beta+1}(K\rho)\right]\mathrm{d}K\right\}\cos\beta\theta \\ &\quad -\frac{4}{\pi}\sum_{\beta=0,1,3}^{\infty}\varepsilon_\beta\left\{\int_0^\infty\frac{F_3(K,\Omega_1,\zeta)\cos\Omega_1(K)\tau}{\Omega_1^2(K)M(K,\Omega_1)}J_\beta(KR)\right. \\ &\quad\times\left.\left[\frac{\beta}{K\rho^2}-KJ_{\beta+1}(K\rho)\right]\mathrm{d}K\right\}\cos\beta\theta.\end{aligned}\tag{50}$$

Since interest here is in the far-field, the terms in equations (48), (49), and (50), involving $1/\rho$ and $1/\rho^2$ may be deleted and thus the dominant, early-arriving, far-field disturbance is given by

$$\begin{aligned}\frac{\mu H\pi u_r}{\sigma_0} &= -\frac{2}{\pi}\sum_{\beta=0,1,3,}^{\infty}\varepsilon_\beta\left\{\int_0^\infty G_3(K\zeta)J_\beta(KR)J_{\beta+1}(K\rho)\,\mathrm{d}K\right\}\cos\beta\theta \\ &\quad+\frac{4}{\pi}\sum_{\beta=0,1,3,}^{\infty}\varepsilon_\beta\left\{\int_0^\infty\frac{KF_3(K,\Omega_1,\zeta)\cos\Omega_1(K)\tau}{\Omega_1^2(K)M(K,\Omega_1)}J_\beta(KR)J_{\beta+1}(K\rho)\,\mathrm{d}K\right\}\cos\beta\theta\end{aligned}\tag{51}$$

$$\begin{aligned}\frac{\mu H\pi u_z}{\sigma_0} &= -\frac{2}{\pi}\sum_{\beta=0,1,3}^{\infty}\varepsilon_\beta\left\{\int_0^\infty G_4(K\zeta)J_\beta(KR)J_\beta(K\rho)\,\mathrm{d}K\right\}\cos\beta\theta \\ &\quad+\frac{4}{\pi}\sum_{\beta=0,1,3,}^{\infty}\varepsilon_\beta\left\{\int_0^\infty\frac{KF_4(K,\Omega_1,\zeta)\cos\Omega_1\tau}{\Omega_1^2M(K,\Omega_1)}J_\beta(KR)J_\beta(K\rho)\,\mathrm{d}K\right\}\cos\beta\theta.\end{aligned}\tag{52}$$

A standard procedure in developing far-field approximations has been to replace the Bessel functions involving ρ in the above equations by their large-argument asymptotic expansions. However, this should be done with some caution, since it is known to lead to errors in other fields (see Fante [15]). Sufficient conditions for the asymptotic expansion of an integral being the integral of the asymptotic expansion of the integrand are (i) the resulting integrals should exist, and (ii) the asymptotic expansion of the integrand should be uniformly valid in the integration parameter over the range of integration (see Erdélyi [16]). Thus if the Bessel functions involving ρ in equations (51) and (52) were to be replaced by their large-argument asymptotic expansions, condition (ii) would be violated, since, for a fixed, large value of ρ, there are values of K in the integration range such that $K\rho$ is small w.r.t. the order of the Bessel functions. Rather than attempt to work with uniformly valid expansions, a possible approach under the above circumstances is as follows:

The integral in the first term on the right-hand side of equation (51) may be written

$$I_1=\int_0^{K_1}L(K)J_{\beta+1}(K\rho)\,\mathrm{d}K+\int_{K_1}^\infty L(K)J_{\beta+1}(K\rho)\,\mathrm{d}K\tag{53}$$

where

$$L(K) = G_3(K\zeta)J_\beta(KR)$$

and the small number K_1 is such that $K_1\rho$ is large w.r.t. the order of the Bessel functions. Thus in the second term on the right-hand side of equation (53), replacement of the Bessel functions by their large-argument asymptotic expansions is permissible. However, the resulting expression is a Fourier integral and so, by the Riemann–Lebesque lemma, the contribution from this source is of order $1/\rho^{\frac{3}{2}}$, which, as will emerge, is of lesser significance than other terms which arise. On expanding $L(K)$ in a Taylor series about $K = 0$, equation (53) gives

$$I_1 = \frac{L(0)}{\rho}\int_0^{K_1\rho} J_\beta(\xi)\,\mathrm{d}\xi + O\left(\frac{1}{\rho^{\frac{3}{2}}}\right)$$
$$= \frac{L(0)}{\rho}\left\{\int_0^\infty J_\beta(\xi)\,\mathrm{d}\xi - \int_{K_1\rho}^\infty J_\beta(\xi)\,\mathrm{d}\xi\right\} + O\left(\frac{1}{\rho^{\frac{3}{2}}}\right)$$

or, replacing J_β in the second integral by its large-argument expansion and evaluating $L(0)$,

$$I_1 = \frac{\sigma}{2\rho}\delta_{0\beta} \tag{54}$$

where σ is Poisson's ratio and $\delta_{0\beta}$ denotes the Kronecker delta. In a similar fashion the first integral on the right-hand side of equation (52) becomes

$$\begin{aligned} I_2 &= \int_0^\infty G_4(K\zeta)J_\beta(KR)J_\beta(K\rho)\,\mathrm{d}K = \frac{G_4(0,\zeta)}{\rho}\delta_{0\beta} + O\left(\frac{1}{\rho^{\frac{3}{2}}}\right) \\ &= 0 + O\left(\frac{1}{\rho^{\frac{3}{2}}}\right). \end{aligned} \tag{55}$$

The second integral on the right-hand side of equation (51) may be written

$$I_3 = \int_0^{K_1\rho} Q_\beta(K)\cos\Omega_1(K)\tau J_{\beta+1}(K\rho)\,\mathrm{d}K + \int_{K_1\rho}^\infty Q_\beta(K)\cos\Omega_1(K)\tau J_{\beta+1}(K\rho)\,\mathrm{d}K$$

where

$$Q_\beta(K) = \frac{KF_3(K,\Omega_1,\zeta)J_\beta(KR)}{\Omega_1^2(K)M(K,\Omega_1)}.$$

Using the large-argument expansion in the second integral one gets

$$I_3 = I_4 + I_5 \tag{56}$$

where

$$I_4 = \int_0^{K_1\rho} Q_\beta(K)\cos\Omega_1(K)\tau J_{\beta+1}(K\rho)\,\mathrm{d}K$$

$$I_5 = \begin{cases} -\dfrac{(\mathscr{R}-\mathscr{I})}{2\sqrt{(\pi\rho)}}\displaystyle\int_{K_1\rho}^\infty \frac{1}{\sqrt{K}}Q_0(K)[\mathrm{e}^{i\rho(K-\Omega_1\tau/\rho)} + \mathrm{e}^{i\rho(K+\Omega_1\tau/\rho)}]\,\mathrm{d}K, & \beta = 0 \quad (57) \\ -\dfrac{(\mathscr{R}+\mathscr{I})\sin(\beta\pi/2)}{2\sqrt{(\pi\rho)}}\displaystyle\int_{K_1\rho}^\infty \frac{1}{\sqrt{K}}Q_\beta(K)[\mathrm{e}^{i\rho(K-\Omega_1\tau/\rho)} + \mathrm{e}^{i\rho(K+\Omega_1\tau/\rho)}]\,\mathrm{d}K, & \beta = 1,3 \quad (58) \end{cases}$$

$\mathscr{I}$ denoting imaginary part.

The integrals I_5 are now in a form to which the method of stationary phase, for large ρ and fixed τ/ρ, is readily applicable and this will be done shortly. Thus, for a uniform treatment, an estimate of I_4, for large ρ and τ is required. So far, this has not been achieved by the authors and the subsequent development hinges on the following physical observation:* since there is a correspondence between wavenumber K and group velocity $c_g(K)$, and hence time of arrival $\rho/c_g(K)$, the integral I_4 describes events between $\tau = \rho/c_g(0)$ and $\tau = \rho/c_g(K_1)$, and for large enough ρ this interval can be made quite small. Thus deleting I_4, i.e., letting K_1 go to zero in I_5, one obtains a response which is valid apart from a small zone in the immediate vicinity of the arrival time corresponding to $K = 0$, which for compressional waves is the arrival time of the plate wave† ρ/c_P, where $c_P^2 = E/\rho'(1-\sigma^2)$, E being Young's modulus. In the subsequent work here, it is to be understood that the values of K and ρ are such that the arrival times involved do not lie in this zone of possible inaccuracy, and henceforth contributions from I_4 are deleted. A similar situation arises in the treatment of the second integral on the right-hand side of equation (52) and again contributions from terms analogous to I_4 are deleted, with the understanding that the subsequent response may lack accuracy in the immediate vicinity of the plate-wave arrival.

Applying the method of stationary phase,‡ and using equations (54) through (58), equations (51) and (52) become

$$\frac{\mu\pi H u_r}{\sigma_0} = \frac{\sigma}{4\rho} - \frac{20}{\rho} V_1(S,\zeta)\left[J_0(SR)\frac{\sin\rho f}{\sin 20f} + \frac{4}{\pi}\frac{\cos\rho f}{\sin 20f}\sum_{\beta=1,3,}^{\infty}\frac{1}{\beta}J_\beta(SR)\cos\beta\theta\right] \tag{59}$$

$$\frac{\mu\pi H u_z}{\sigma_0} = \frac{20}{\rho} V_2(S,\zeta)\left[\frac{\cos\rho f}{\cos 20f}J_0(SR) - \frac{4}{\pi}\frac{\sin\rho f}{\cos 20f}\sum_{\beta=1,3,}^{\infty}\frac{1}{\beta}J_\beta(SR)\cos\beta\theta\right] \tag{60}$$

where

$$f = \Omega_1(S)\left[\frac{1}{\Omega_1'(S)} - \frac{S}{\Omega_1(S)}\right]$$

S is a point of stationary phase, i.e., a root of

$$c_g(S) = \rho/\tau$$

and the functions V_1 and V_2, which have been computed by Miklowitz [6], are given by

$$V_1(S,\zeta) = \frac{1}{10}\left[\frac{S\Omega_1'(S)}{|\Omega_1''(S)|}\right]^{\frac{1}{2}} \frac{F_3(S,\Omega_1,\zeta)}{2\Omega_1^2(S)M(S,\Omega_1)}\sin\rho f$$

$$V_2(S,\zeta) = -\frac{1}{10}\left[\frac{S\Omega_1'(S)}{|\Omega_1''(S)|}\right]^{\frac{1}{2}} \frac{F_4(S,\Omega_1,\zeta)}{2\Omega_1^2(S)M(S,\Omega_1)}\cos\rho f.$$

Some numerical calculations based on equations (59) and (60), are presented in Figs. 2 and 3. The interest here is in the effects of non-axisymmetry, and so the computations were performed at a fixed station ($\rho = 1000$, $\zeta = 1$), for several discrete values of S (i.e., time).

* Integrals such as I_4 could of course be treated numerically.

† On the basis of the first three modes of the Rayleigh–Lamb frequency equation, the plate wave is the earliest arriving disturbance in the far-field.

‡ Noting that there is no contribution from terms involving $K+\Omega_1\tau/\rho$, since for the portion of the spectrum under consideration, this factor does not have points of stationary phase.

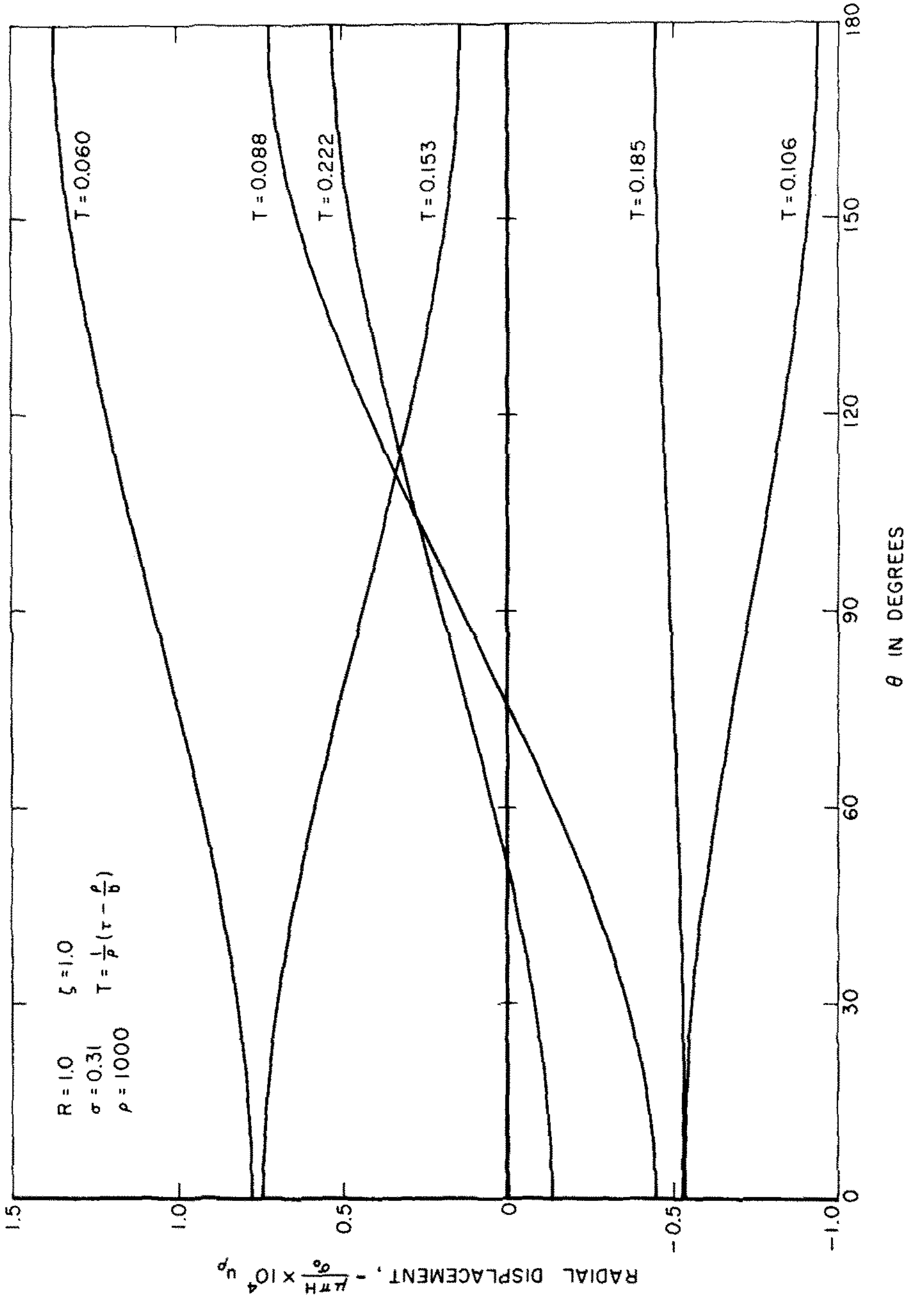

FIG. 2. Radial displacement as a function of θ.

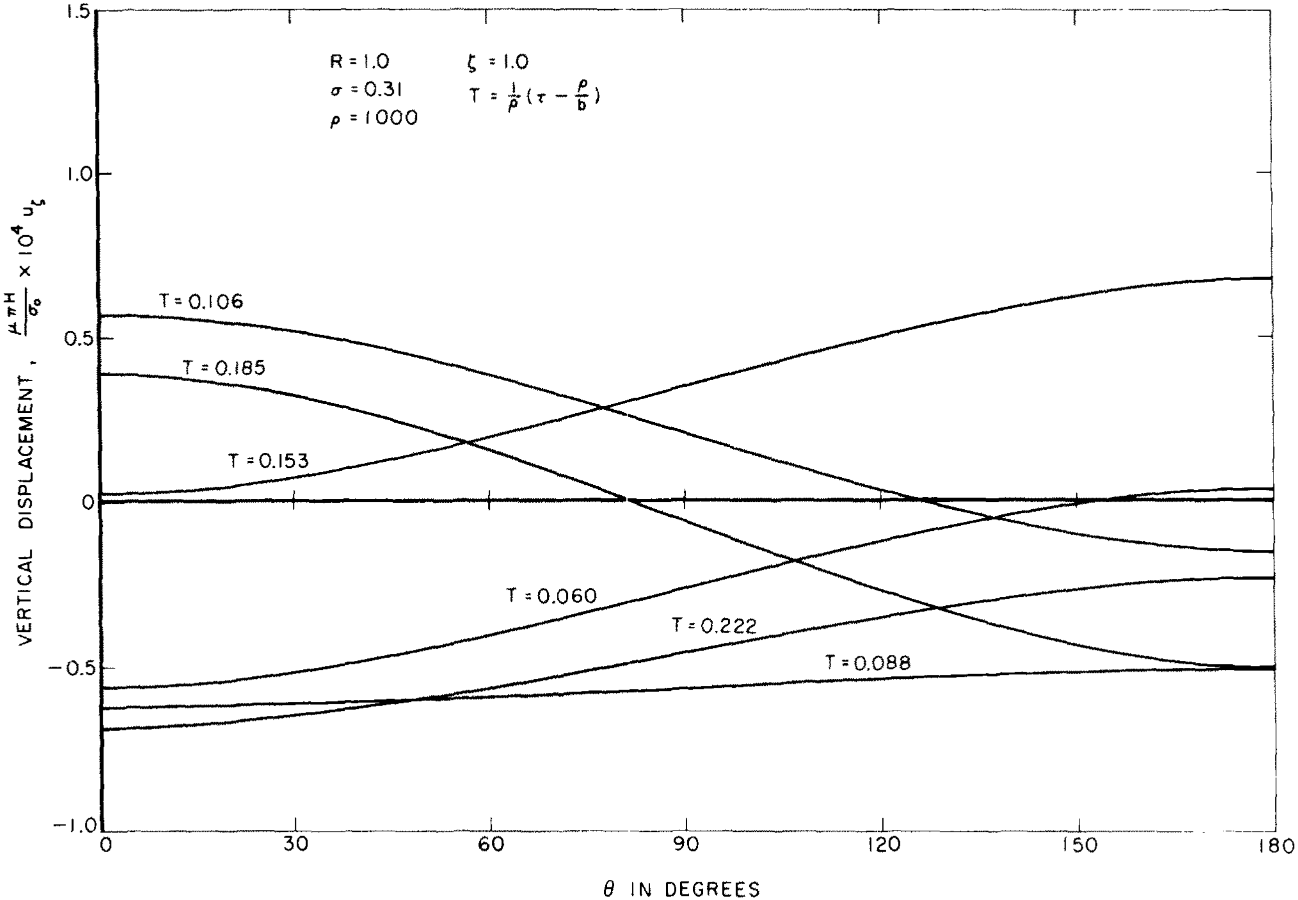

FIG. 3. Vertical displacement as a function of θ.

The other parameters involved were taken as $R = 1$, $\sigma = 0{\cdot}33$. The above values, and the choices of S, are such that the times involved are larger than the arrival time of the plate wave, and moreover, very rapid convergence of the infinite series takes place.

Figure 2 gives the dynamic radial displacement (i.e., u_r as given by equation (59), minus the static solution $\sigma/4\rho$) as a function of θ, for $0 \leq \theta \leq 180°$—the only range required, since u_r is an even function of θ. An interesting feature of the response is the time oscillation that is developed, as evidenced by the fact that for $T^* = 0{\cdot}060$, the minimum amplitude occurs at $\theta = 0°$, and the maximum occurs at $\theta = 180°$, whereas for $T = 0{\cdot}106$, the minimum amplitude occurs at $\theta = 180°$, and the maximum occurs at $\theta = 0°$. The figure shows that as time progresses the process is cyclic, the response for some times (i.e., $T \approx 0{\cdot}185$) being axisymmetric.

Figure 3 gives the vertical displacement as a function of θ. It is seen that the same overall features as noted above arise here also, i.e., a disturbance in which the maximum and minimum amplitudes are either at $\theta = 0°$ or $\theta = 180°$, depending on the value of the time involved.

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Абстракт—Используя линейную теорию упругости, определяются формальные решения для общего класса не симметрических, нестационарных задач распределения упругих волн, вызваных бесконечной, изотропной, упругой пластинкой. Получается явное преобразование для частной задачи внезапно приложения нормальной, полукольцевой нагрузки на поверхностях пластинки. Представляются некоторые широкие численные результаты, основаны на приближении ниской частоты и большой длины волны.

* $T = (1/\rho)(\tau - \rho/b)$, where $b = c_P/c_S$.